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Theoretical and Experimental Study of the Resonant Frequency of a Cylindrical Dielectric Resonator

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***Abstract*—A rigorous modal method is described for calculating the resonant frequencies of a circular cylindrical dielectric rod placed between two perfectly conducting plates. Comparisons of the numerical results with those obtained from another rigorous theory developed at the same time by one of the authors show an accuracy better than 10^{-4}. Comparison with experimental data shows generally a very good agreement.**

I. INTRODUCTION

During the last decade, dielectric resonators met an increasing interest, due to the development of temperature-stable materials [1], [2].

Several methods have been proposed in order to solve the problem of determining resonant frequencies. The earliest papers dealt with simple devices like a sphere, or a cylinder between metallic planes [3]–[5]. Practical devices require more intricate calculations, and solutions are often approximate [6]–[11].

The theoretical method described here can be considered as an extension of the rigorous study by Hakki and Coleman [4] in the case where the distances between the dielectric rod and the metallic plates are different from zero. Space is divided into two complementary cylindrical regions where the field is expressed in the form of two modal expansions with unknown coefficients. The matching between these two expansions leads to an infinite set of homogeneous linear equations. The resonant frequency is obtained by looking for the zero of the determinant of the truncated matrix.

A comparison is made with another rigorous theory, the differential theory, quite different in nature, and the relative discrepancy never exceeds 10^{-4} when the two methods can be implemented. Even though the same conclusion cannot be drawn for the comparison with experimental data, the agreement is very good and appears to be satisfactory, taking into account the uncertainties about the actual experimental parameters and the influence of the finite conductivity of the two plates. Thanks to the great precision of the computer code, we are able to show that the resonant frequency may be estimated very simply from an equivalence rule, provided the air gaps are small.

II. THEORY

A. Basic Equations

We deal with the circular cylindrical rod represented in Fig. 1, with permeability μ_0, real relative permittivity ϵ, length h_2, and radius R. It is placed at distances h_1 and h_3 from two perfectly conducting plates parallel to the Oxy plane. We denote by $l = h_1 + h_2 + h_3$ the distance between these two plates.

The aim of this study is to compute the fundamental TE resonant frequency. The θ component $F(r, z)$ of the electric field, which is independent of θ, satisfies the following equation:

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \left(k^2(r,z) - \frac{1}{r^2}\right)F + \frac{\partial^2 F}{\partial z^2} = 0 \tag{1}$$

where $k^2(r,z)$ is the wavenumber which is equal to $k_0^2 = (2\pi/\lambda)^2$ in the air and to $k_0^2\epsilon$ in the rod.

On top of that, F must satisfy the following boundary conditions:

$$\begin{aligned} &F(r,0) = F(r,l) = 0\\ &F \text{ and } \frac{\partial F}{\partial z} \text{ are continuous for } z = h_1 \text{ and } z = h_1 + h_2\\ &F, \frac{\partial F}{\partial r} \text{ and consequently } \frac{\partial (rF)}{\partial r} \text{ are continuous for } r = R. \end{aligned} \tag{2}$$

Of course, $F(O, z)$ must vanish since F is a θ component, and $F(r, z)$ satisfies a radiation condition when $r \to \infty$; in other words, the field must decay exponentially outside the resonator.

B. Modal Expansions

Space is divided into two regions Ω_{ext} $(r > R)$ and Ω_{in} $(r < R)$. For both regions, we establish that the field may be expanded in series.

In Ω_{ext}, $k^2(r, z)$ remains constant and equal to k_0^2. From (1)

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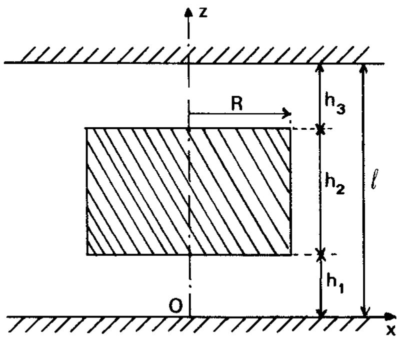

Fig. 1. The cylindrical dielectric resonator.

and (2), it can be written rigorously that

$$F(r,z)=\sum_{n=1}^{\infty} B_n K_1(\mu_n r)t_n(z) \quad (3)$$

with

$$t_n(z)=\sin\left(\frac{n\pi}{l}z\right)$$

$$\mu_n=\left(n^2\pi^2/l^2-k_0^2\right)^{1/2} \text{ or } -i\left(k_0^2-\frac{n^2\pi^2}{l^2}\right)^{1/2}$$

and K_1 denotes the modified Bessel function. In practice, we look for nonradiating modes, for which all the μ_n are real.

Since Ω_{in} is not homogeneous, it is much more difficult to find the relevant modal expansion in this region. Remarking that in fact $k^2(r,z)$ is independent of r in this region, we can write (1) in the form

$$D_r F + D_z F = 0 \quad (4)$$

where D_r and D_z are operators acting on functions of r and z, respectively.

Since D_z is a self-adjoint operator, F can be expanded in series using the system of orthogonal eigenfunctions $v_n(z)$ of D_z with real eigenvalues η_n [12], and it turns out that v_n may be expressed in the form

$$\text{if } 0 \leqslant z \leqslant h_1, \quad v_n=a_n \sin(\alpha_n z) \quad (5)$$

$$\text{if } h_1 \leqslant z \leqslant h_1+h_2, \quad v_n=b_n\sin(\beta_n z)+c_n\cos(\beta_n z) \quad (6)$$

$$\text{if } h_1+h_2 \leqslant z \leqslant l, \quad v_n=d_n\sin[\alpha_n(z-l)] \quad (7)$$

with

$$\alpha_n=\left(k_0^2-\eta_n\right)^{1/2} \text{ or } i\left(\eta_n-k_0^2\right)^{1/2}$$

$$\beta_n=\left(k_0^2\epsilon-\eta_n\right)^{1/2} \text{ or } i\left(\eta_n-k_0^2\epsilon\right)^{1/2}.$$

From the boundary conditions for v_n and v_n' at $r=h_1$ and $r=h_1+h_2$, we obtain after tedious calculations

$$\phi_n+\psi_n+\beta_n h_2=q\pi \quad (8)$$

with

$$tg\,\phi_n=\frac{\beta_n}{\alpha_n}tg(\alpha_n h_1)$$

$$tg\,\psi_n=\frac{\beta_n}{\alpha_n}tg(\alpha_n h_3).$$

It is explained in the Appendix how the solutions of (8) may be obtained in a systematic manner. Now, let us show a fundamental property of these solutions: the real eigenvalues η_n are less than $k_0^2\epsilon$. With this aim, we evaluate the integral I_n defined by

$$I_n=\int_0^l v_n''\bar{v}_n\,dz. \quad (9)$$

First, we can notice that $v_n''\bar{v}_n=(v_n'\bar{v}_n)'-|v_n'|^2$, and since $v_n(0)=v_n(l)=0$

$$I_n=-\int_0^l |v_n'|^2\,dz. \quad (10)$$

On the other hand, I_n may be expressed by replacing v_n'' by $(\eta_n-k^2(r,z))v_n$

$$I_n=\int_0^l\left(\eta_n-k^2(r,z)\right)|v_n|^2\,dz. \quad (11)$$

From (10), we deduce that I_n is always negative; then we derive from (11) that (η_n-k^2) must be negative, at least on a certain part of the interval $(0,l)$. Obviously, that is impossible if $\eta_n>k_0^2\epsilon>k_0^2$, therefore

$$\eta_n<k_0^2\epsilon. \quad (12)$$

Finally, in Ω_{in}, the field may be rigorously represented by the expansion

$$F(r,z)=\sum_{n=1}^{\infty} A_n J_1(\xi_n r)v_n(z) \quad (13)$$

with

$$\xi_n=(\eta_n)^{1/2} \text{ or } i(-\eta_n)^{1/2}$$

J_1 being the Bessel function of the first kind.

C. Matching the Two Modal Expansions

We match the modal expansion (3) and (13) by writing the continuity of F and $d(rF)/dr$ for $r=R$. The identities $d(xJ_1(x))/dx=xJ_0(x)$ and $d(xK_1(x))/dx=-xK_0(x)$ give

$$\sum_{n=1}^{\infty} B_n K_1(\mu_n R)t_n(z)=\sum_{m=1}^{\infty} A_m J_1(\xi_m R)v_m(z) \quad (14)$$

$$-\sum_{n=1}^{\infty}\mu_n B_n K_0(\mu_n R)t_n(z)=\sum_{m=1}^{\infty}\xi_m A_m J_0(\xi_m R)v_m(z). \quad (15)$$

Projecting the two members of (14) and (15) on an arbitrary function $v_m(z)$, and taking into account the orthogonality of the v_m, yield

$$\forall m,\ \sum_{n=1}^{\infty} B_n K_1(\mu_n R)\langle v_m,t_n\rangle=A_m J_1(\xi_m R)\langle v_m,v_m\rangle \quad (16)$$

$$-\sum_{n=1}^{\infty}\mu_n B_n K_0(\mu_n R)\langle v_m,t_n\rangle=\xi_m A_m J_0(\xi_m R)\langle v_m,v_m\rangle \quad (17)$$

where

$$\langle v_m,t_n\rangle=\int_0^l \bar{v}_m(z)t_n(z)\,dz.$$

Eliminating A_m between the two above equations, it emerges

$$\forall m,\ \sum_{n=1}^{\infty} B_n[\mu_n K_0(\mu_n R)J_1(\xi_m R)+\xi_m J_0(\xi_m R)K_1(\mu_n R)]\langle v_m,t_n\rangle=0 \quad (18)$$

TABLE I
CONVERGENCE OF THE RESONANT WAVELENGTH FOR $h_1 = 0$, $h_2 = 13.37$, $R = 8.995$, $\epsilon = 34.61$, WHEN THE SIZE N OF THE SYSTEM IS INCREASED IN THE MODAL METHOD. THE LAST LINE GIVES THE RESULT OBTAINED FROM THE DIFFERENTIAL METHOD

N	Resonant wavelength for h_3 = 9.03	Resonant wavelength for h_3 = 33.03
1	104.113	110.941
2	102.999	105.237
3	102.898	103.873
4	102.893	103.978
5	102.879	104.035
6	102.878	103.987
7	102.877	103.947
9	102.876	103.943
11	102.876	103.933
17	102.876	103.929
21		103.929
Diff.	102.873	103.923

where the scalar product $\langle v_n, t_m \rangle$ is given by

$$\langle v_n, t_m \rangle = S_{nm}\left\{\frac{m\pi}{l}\left[\sin(\phi_n)\cos\left(\frac{m\pi}{l}h_1\right) + (-1)^m \sin(\psi'_n)\cos\left(\frac{m\pi}{l}h_3\right)\right] - \beta_n\left[\cos(\phi_n)\sin\left(\frac{m\pi}{l}h_1\right) + (-1)^m\cos(\psi'_n)\sin\left(\frac{m\pi}{l}h_3\right)\right]\right\} \quad (19)$$

with

$$\psi'_n = -\phi_n - \beta_n h_2 = \psi_n - q\pi.$$

D. Numerical Implementation

To find the resonant frequency we first put the value of $\langle v_n, t_m \rangle$ given by (19) into (18), then we truncate the homogeneous system of linear equations so obtained. Then the frequency is varied and we look for the zero of the determinant of the system using a *regula falsi* method. Table I shows the convergence of the numerical results when the size N of the system is increased. One can see that $N = 5$ is enough to obtain the asymptotic result (102.873) to within 10^{-4}, for $h_3 = 9.03$. This conclusion no longer holds for $h = 33.03$ where $N = 7$ is necessary to obtain this precision. These results have been compared with those obtained from a rigorous differential theory developed by one of the present authors (P.V.), which is to be published in *Applied Physics A*. It is interesting to notice that the two formalisms, quite different from a theoretical point of view, agree to within 3.10^{-5} for $h_3 = 9.03$, or to within 6.10^{-5} for $h_3 = 33.03$. It is also worth noting that the asymptotic value given by the modal theory is likely to be the best one. Indeed, for reasons of computation time, the convergence of the right-most digit given by the differential theory has not been observed. The great precision is all the more remarkable since the computation is made with a HP 1000 F computer, with only six significant digits. It appears that N must increase while $(h_1 + h_3)/h_2$ is increased. On the whole, the execution time takes a few seconds.

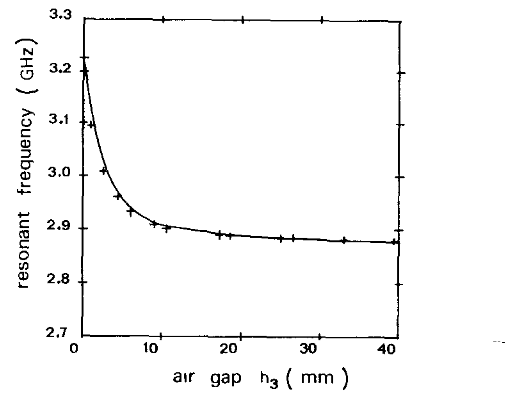

Fig. 2. Comparison between theoretical and experimental results for a resonator with $R = 8\,995$ mm, $h_1 = 0$, $h_2 = 13.37$ mm, $\epsilon = 34.61$.

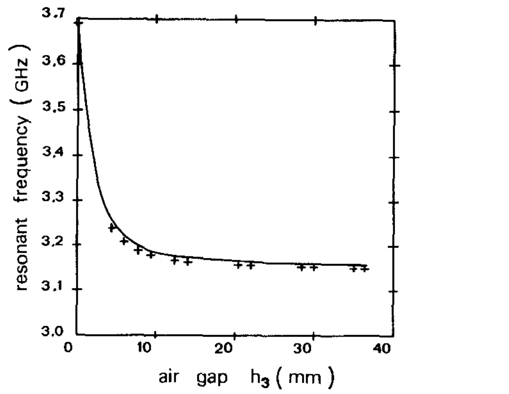

Fig. 3. The same as Fig. 2 but $R = 8.99$ mm, $h_1 = 0$, $h_2 = 9.99$, $\epsilon = 35.29$.

E. Comparison with Other Theories

When $h_1 = h_3 = 0$, our formalism tends to the one used by Hakki and Coleman [4]. Indeed, in this case, the functions $v_n(z)$ and $t_n(z)$ become proportional since $\phi_n = p\pi$ and $\psi_n = p\pi$, according to (8). It turns out that $\langle v_n, t_m \rangle$ is nil, except for $n = m$, and the sum in the left-hand member of (18) reduces to a single term $n = m$, ξ_m being equal to $(k_0^2\epsilon - m^2\pi^2/l^2)^{1/2}$ or $i(m^2\pi^2/l^2 - k_0^2\epsilon)^{1/2}$. The fundamental resonant frequency is obtained by solving the equation obtained for $m = 1$, i.e., the basic equation given by Hakki and Coleman [4]. The equations for $m = 2, 3, \cdots$ provide the resonant frequencies of higher modes.

It is very important to notice that, as soon as h_1 or h_3 differs from zero, the calculation of the fundamental resonant frequency requires taking into account all the values of m, at least from a theoretical point of view. In other words, the non-null value of h_1 or h_3 creates a coupling between all the terms of the series in (13). Recently, Bonetti and Atia [9] proposed a method for a closely related problem. In this method, the field for $r < R$ is only described by the first term of the series contained in (13). This theory appears to be questionable from a theoretical point of view and in practice, it is obvious, on Table I, that this approximation, which is close to the approximate solution obtained with $N = 1$, cannot provide the same accuracy as a rigorous method, especially if h_1 or h_3 is not negligible.

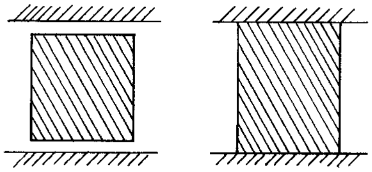

Fig. 4. The resonant frequencies of the two resonators are very close to each other. (a) Case I. (b) Case II.

III. Comparison Between Experimental and Theoretical Results

The measurement cell is composed of two copper plates (150× 150 mm). In the lower plate, two symmetric loops perform coupling from the microwave generator and to the detector. Very low coupling (0,01) avoids perturbation of the natural resonant frequency. The dielectric rod is laid on the lower plate, between the loops ($h_1 = 0$), and the upper plate can be set in order to vary h_3.

The value of the permittivity is deduced from the case $h_3 = 0$, using the theory of Hakki and Coleman [4].

Figs. 2 and 3 show a comparison between theoretical and experimental results, for two different resonators. The agreement obtained in Fig. 2 is very good, generally better than 10^{-3} in relative value. On the other hand, the discrepancy observed in Fig. 3 is higher, of the order of 3.10^{-3}. We conjecture that the main origin of the discrepancy must be found in heterogeneousness of the dielectric material. Indeed, small spatial variations of the permittivity related to density or chemical composition fluctuations imply significant frequency shifts. This result means that consistency of the permittivity is an important parameter in elaborating devices such as filters.

IV. Equivalence Rule

From our numerical computation, we have deduced the following empirical rule:

If h_1/h_2 and h_3/h_2 are small, the resonant frequency is not changed, to the first order when the air gaps between the dielectric and the two plates are filled with a dielectric of the same permittivity ϵ.

This rule, outlined in Fig. 4, can be easily demonstrated by remarking that, when h_1 and h_3 are small, (8) becomes

$$\tan\phi_n \simeq \beta_n h_1 \tag{20}$$

$$\tan\psi_n \simeq \beta_n h_3 \tag{21}$$

$$\tan(\beta_n h_2) = -\tan(\phi_n + \psi_n) \simeq -\beta_n(h_1 + h_3). \tag{22}$$

We deduce that $\tan(\beta_n h_2)$ is small, thus

$$\beta_n h_2 \simeq -\beta_n(h_1 + h_3) + p\pi \tag{23}$$

and finally

$$\beta_n = \frac{p\pi}{l}. \tag{24}$$

These values of β_n are exactly those obtained for the resonator corresponding to case II of Fig. 4. Moreover, it is easy to show that, when h_1 and h_3 are small, $\langle v_n, t_m \rangle$ given by (19) is nil to the first order. Finally, we can conclude that (18) is the same, to the first order, in the two cases of Fig. 4.

This empirical rule is numerically verified in Fig. 5. We have plotted the resonant frequency of a dielectric resonator having $h_1 = 0$, when h_3 increases, l being constant. It is worth noting that the resonant frequency is a quadratic function of $h_3 = l - h_2$ at the right-hand side of the figure. This entails that a relative variation of 2.10^{-1} of h_2 only produce a corresponding variation of 10^{-2} of the resonant frequency, provided l is kept constant.

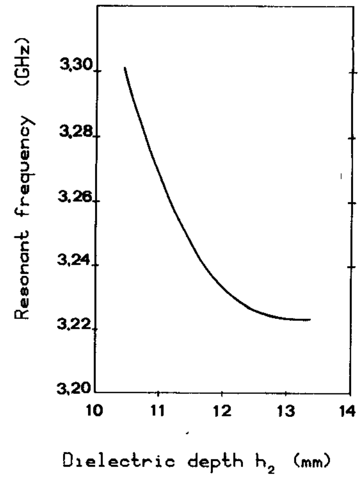

Fig. 5. Resonant frequency of a dielectric resonator as a function of h_2 for $h_1 = 0$, $h_2 + h_3 = l = 13.37$, $R = 8.995$, $\epsilon = 34.61$.

This rule appears to be very interesting in practice since it permits the use of the simple calculation proposed by Hakki and Coleman to know the resonant frequency of a dielectric resonator with a good accuracy, provided h_1/h_2 and h_3/h_2 are small.

V. Conclusion

The rigorous method presented in this paper allows us to calculate with great precision the resonant frequency of the fundamental TE mode of a dielectric resonator. The accuracy of this method is better than the consistency of experimental results, so that the agreement of this "modal" method with a quite different "differential" method is a more meaningful criterion than comparison with experiments.

Appendix

We have to solve (8), which can be written in the form

$$F(u) = \tan(\phi_n + \psi_n) + \tan(\beta_n h_2) = 0. \tag{A1}$$

The left-hand member will be considered as a function of $u = \beta_n$, which is positive, as seen in Section II, with

$$\tan\phi_n = \frac{u}{v}\tan(h_1 v) \tag{A2}$$

$$\tan\psi_n = \frac{u}{v}\tan(h_3 v) \tag{A3}$$

$$v = \left(u^2 + k_0^2(1-\epsilon)\right)^{1/2} \text{ or } i\left(k^2(\epsilon-1) - u^2\right)^{1/2}. \tag{A4}$$

It is worth noting that the right-hand member of (A2) and (A3) is always real, in the interval $u \in (0, k_0^2(\epsilon - 1))$ where v is imaginary.

From cumbersome calculations, it can be shown that ϕ_n and ψ_n are monotonous (increasing) functions of u. So $\tan(\phi_n + \psi_n)$ is also an increasing function of u, and since $\tan(h_2 u)$ satisfies the same property, $F(u)$ is monotonous.

This fundamental property enables us to assert that the number of non-null solutions of (A1) between 0 and the abscissa u_n of the nth asymptote ($F(u_n) = \infty$) is equal to $n - 1$. A part of these asymptotes are analytic and given by $\tan(h_2 u) = 0$, i.e.,

$$u = (2p+1)\frac{\pi}{2h_2}, \qquad \text{with } p \geq 0. \tag{A5}$$

The complementary subset of asymptotes is given by $\tan(\phi_n + \psi_n) = \infty$, or, developing the tangent and using (A2) and (A3)

$$G(u) = \left(\frac{\tan vh_1}{v}\right)\left(\frac{\tan vh_3}{v}\right) = \frac{1}{u^2}. \tag{A6}$$

The left-hand member of (A6) is the product of two monotonous functions but is not monotonous. However, a careful examination of $G(u)$ shows that between 0 and the mth asymptote u'_m of $G(u)$, the number of solutions of (A6) is equal to m.

Now, the values of u'_m are analytic and given by

$$vh_1 = (2q+1)\frac{\pi}{2} \tag{A7}$$

or

$$vh_3 = (2q'+1)\frac{\pi}{2}. \tag{A8}$$

Finally, in an interval $(0, u'_m)$, we are able to determine the number m of non-analytic asymptotes of $F(u)$, thus to derive the total number $n = m + m'$ of asymptotes, m' being the number of analytic asymptotes given by (A5) between 0 and u'_m.

To compute the successive zeros of $F(u)$, we calculate $F(u)$ at a certain number P of equidistant points between 0 and u'_m. We detect the approximate positions of the zeros (when the sign of $F(u)$ goes from a negative to a positive value) and of the asymptotes (the opposite). The number n' of computed asymptotesis compared to $n = m + m'$. If $n' < n$, the computation is started again, with $2P$ points, then $4P$ points, etc, as long as n' is not equal to n. So, we know that all the asymptotes have been detected, and it suffices to verify that a zero of $F(u)$ has well been found between two consecutive asymptotes to be sure that no zero has been forgotten until the last asymptote. A second step of the computation is to enhance the precision on the zeros by using the method of the secant, starting from the approximate values.

As regards the numerical implementation, this very effective systematic method appears to be necessary since the numerical exploitation has shown that $F(u)$ can have zeros and asymptotes very close to each other. Therefore, a rough search for the zeros often misses some of them, which is catastrophic for the final result.

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Moment-Method Solutions and SAR Calculations for Inhomogeneous Models of Man with Large Number of Cells

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***Abstract*—This paper describes an iterative band approximation method (BAM) that is useful for solution of large matrix equations where the elements of the matrix decrease in magnitude with increasing distance from the diagonal. The method involves the inversion of a band about the diagonal which is used to obtain a first estimate of the solution. This estimate, along with the remaining elements in the matrix above and below the band, is used to iterate to the final solution. Due to the substantial reduction in the size of the matrix which is actually inverted, the method has been applied to the solution of full complex matrix equations involving up to 1698 unknowns. BAM is used to obtain distributions of EM energy absorption for man models with 180–1132 cells.**

I. Introduction

To understand the biological effects of electromagnetic fields, it is necessary to quantify the whole-body absorption and its distribution for the various irradiation conditions. Moment-method solutions with inhomogeneous man models have been used to obtain the distributions of time-rates of absorbed energy (specific absorption rates (SAR's)) for free-space irradiation [1]–[3], for a human in contact with and slightly removed from a ground plane and in the presence of metallic corner reflectors [4]. Combined with the plane-wave-spectrum approach to prescribe the incident fields, moment-method solutions have also been used to obtain SAR distributions for leakage-type (uncoupled) near-field exposure conditions such as those from RF sealers, etc. [5]. In fact, in spite of the claims made for other numerical approaches such as finite-element methods, etc., the moment-method is the only successful procedure used at the present time to obtain SAR distributions for inhomogeneous models of biological bodies.

Most of our work has used a block model of man using 180 cubical cells of various sizes arranged for a best fit of the contour on diagrams of the 50th percentile standard man [3]. With considerably larger computation times, we have also described solutions in which a total of 340 cells were used to provide a finer detail of energy deposition in the head and neck allowing us to pinpoint the frequency region for head resonance [6]. The procedures used in the past have required the use of full complex matrices $3N \times 3N$ in dimension for a model with N cells. Effi-

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